Various Facets of Chalker-Coddington network model

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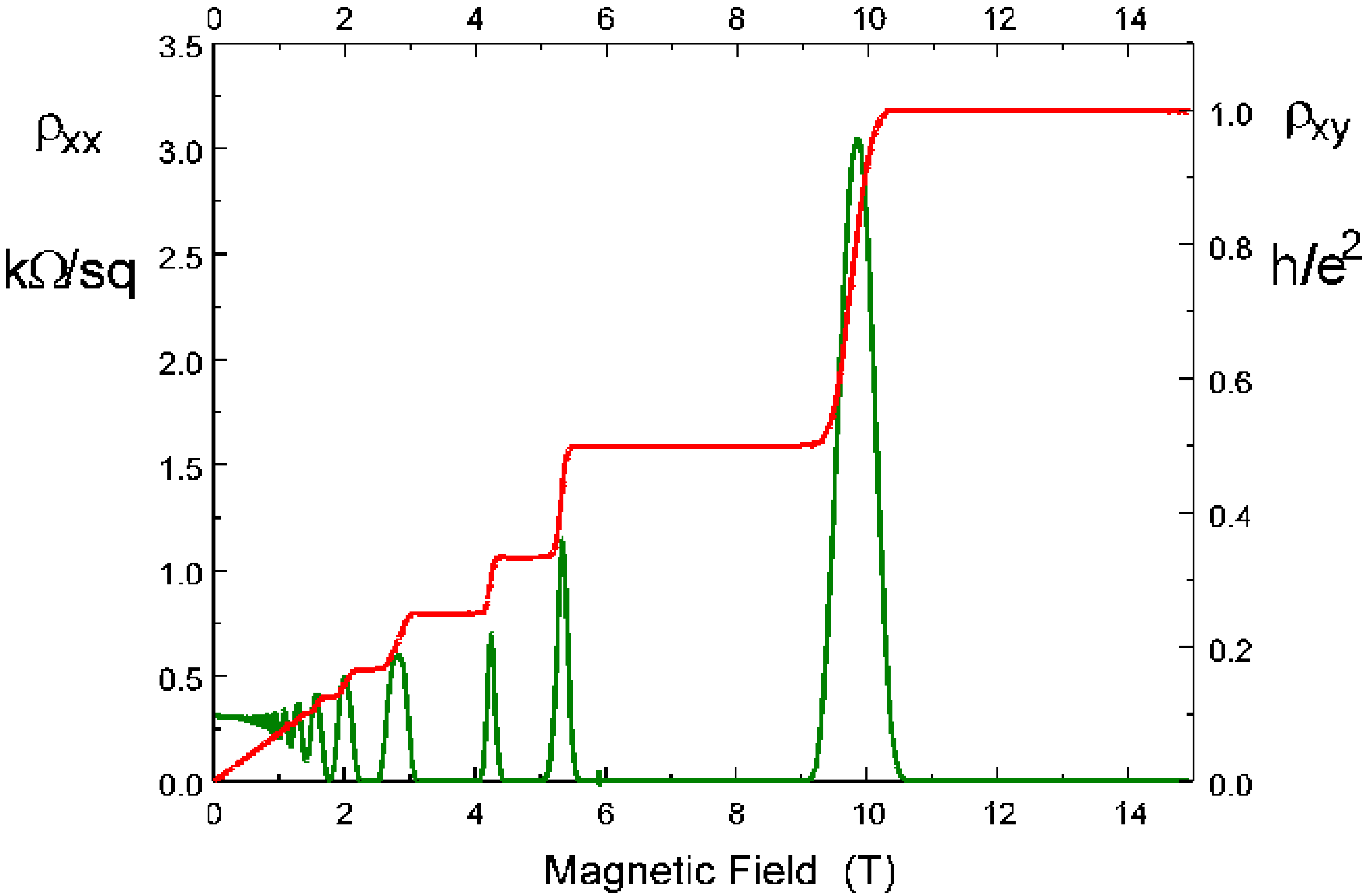
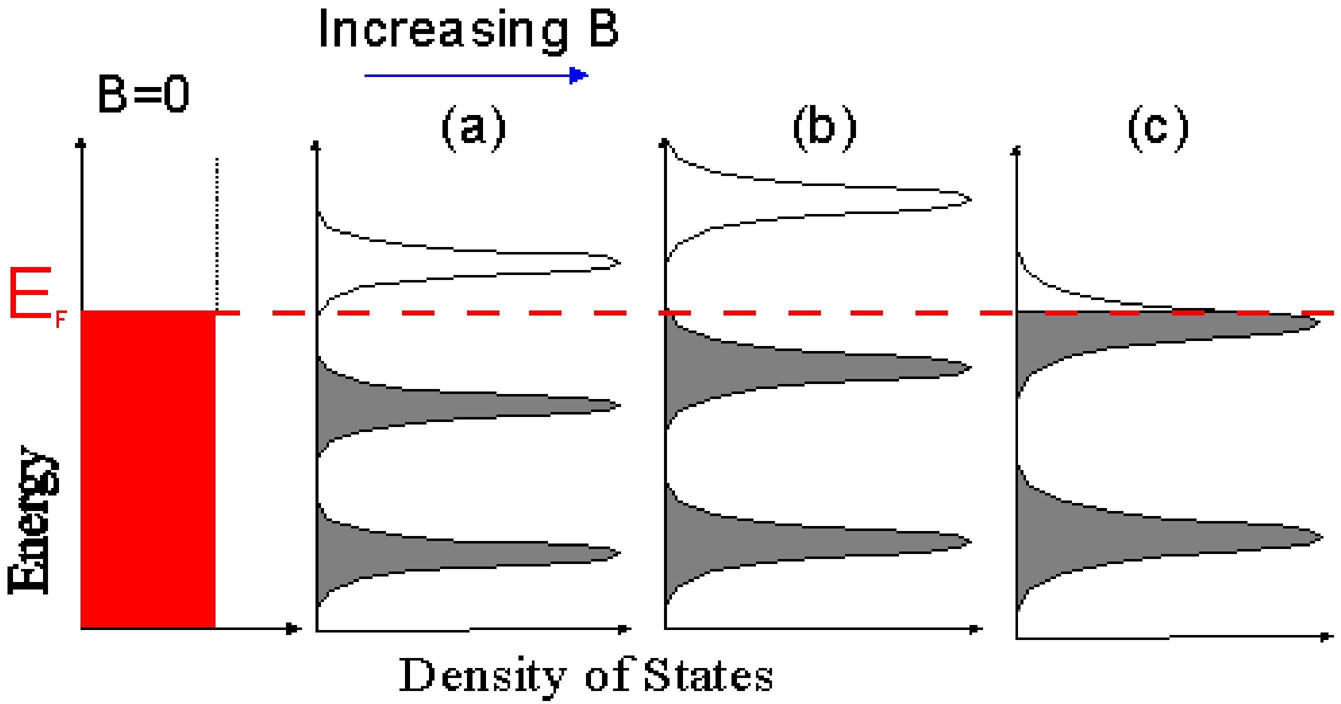
Context

- **Integer quantum Hall effect**
- **Semiclassical picture**
- **Chalker-Coddington network model**
- **Various applications**
  - Inter-plateaux transitions
  - Floating of extended states
  - New symmetry classes in dirty superconductors
  - Effect of nuclear magnetization on QHE



ρxx
kΩ/sq
ρxy
h/e2
0
2
4
6
8
10
12
14
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
1.0
0.8
0.6
0.4
0.2
0.0
Magnetic Field (T)

Bremen



Inter-plateaux transition is a critical phenomenon

Semiclassical picture: strong magnetic field *B* and slowly varying random potential *V(r)*

Criterion: $\ell \nabla V << \hbar \omega_c$ **or** $\ell << \xi_{cor}$

Where magnetic length $\ell = \sqrt{\frac{\hbar c}{eB}}$

cyclotron frequency $\omega_c = \frac{eB}{mc}$

correlation length of potential ξ_{cor}

Hamiltonian of a 2D electron in a perpendicular magnetic filed and random potential

$$H = \frac{1}{2m}\left[\hat{\mathbf{p}} + \frac{e}{c}\hat{\mathbf{A}}(\mathbf{r})\right]^2 + V(\mathbf{r})$$

Standard change of variables

$$\hat{\pi} = \hat{\mathbf{p}} - \frac{e}{c}\hat{\mathbf{A}}, \quad \hat{X} = x - \frac{c}{eB}\hat{\pi}_y, \quad \hat{Y} = y + \frac{c}{eB}\hat{\pi}_x$$

Equations of motion for the guiding center coordinates

$$\dot{X} = \frac{i}{\hbar}\left[H, X\right] = \frac{c}{eB}\frac{\partial V}{\partial y}, \quad \dot{Y} = \frac{i}{\hbar}\left[H, Y\right] = \frac{c}{eB}\frac{\partial V}{\partial x}$$

$$\left[\hat{\pi}_x,\hat{\pi}_y\right]=\frac{\hbar eB}{ic} \quad \Rightarrow \quad \delta\pi_x\approx\delta\pi_y\approx\sqrt{\frac{\hbar eB}{c}} \quad \Rightarrow$$

$$x-X=\frac{c}{eB}\hat{\pi}_y\approx\ell$$

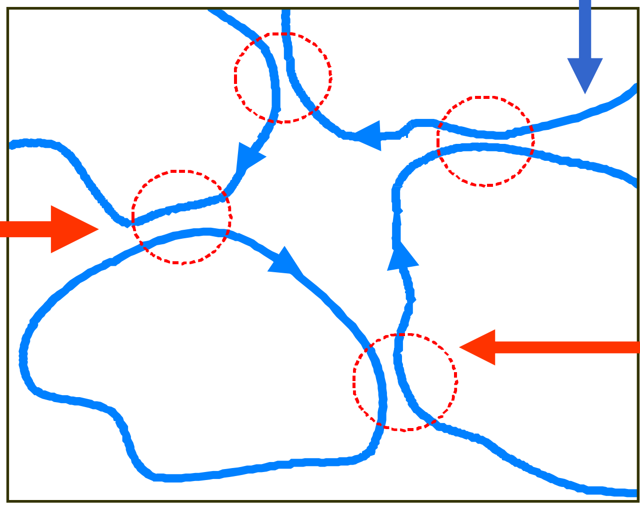
In the limit of **strong magnetic field**

$$X\rightarrow x,\quad Y\rightarrow y$$

$$\frac{dV}{dt}\approx\frac{\partial V}{\partial X}\dot{X}+\frac{\partial V}{\partial Y}\dot{Y}=\frac{c}{eB}\frac{\partial V}{\partial X}\frac{\partial V}{\partial Y}-\frac{c}{eB}\frac{\partial V}{\partial Y}\frac{\partial V}{\partial X}=0$$

Therefore electron moves along **lines of constant potential**

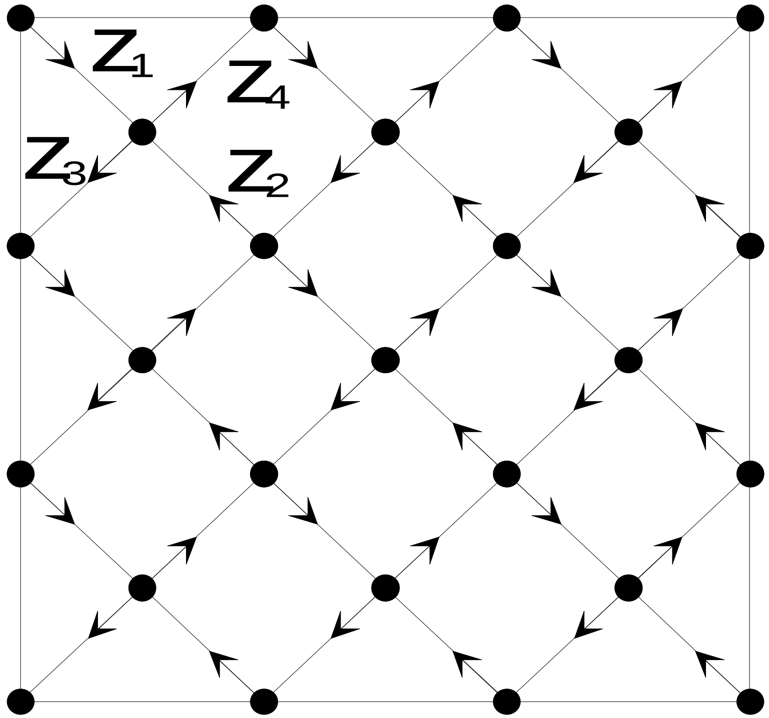
Scattering in the vicinity of the saddle point potential



Transmission probability

$$T=\frac{1}{1+\exp(-\pi\varepsilon)}$$

The network model of Chalker and Coddington. Each node represents a saddle point and each link an equipotential line of the random potential (Chalker and Coddington; 1988)



$$\begin{pmatrix} Z_1 \\ Z_3 \end{pmatrix} = M \begin{pmatrix} Z_4 \\ Z_2 \end{pmatrix}$$

$$M = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} e^{i\varphi_3} & 0 \\ 0 & e^{i\varphi_4} \end{pmatrix}$$

Fertig and Halperin, PRB 36, 7969 (1987)

Exact transmission probability through the saddle-point potential $V_{SP} = U(-x^2 + y^2) + V_0$

$$T = \frac{1}{1+\exp(-\pi\varepsilon)}$$

$$\varepsilon \equiv (E - (n+1/2)E_2 - V_0)/E_1$$

$E_2 \approx \hbar\omega_c$ $\quad E_1 \approx \hbar \frac{2U}{m\omega_c}$ **for strong magnetic fields**

For the network model

$$T = \frac{1}{\cosh^2\theta}$$

$$\varepsilon = -\frac{2}{\pi}\ln(\sinh\theta)$$

Total transfer matrix T of the system is a result of N iterations. Real parts of the eigenvalues are produced by diagonalization of the product

$$(T^{\dagger}T)^{1/2} = \begin{pmatrix} e^{\lambda_1 N} & & & & & & & \\ & e^{\lambda_2 N} & & & & & & \\ & & \ddots & & & & & \\ & & & e^{\lambda_{M/2} N} & & & & \\ & & & & e^{-\lambda_{M/2} N} & & & \\ & & & & & \ddots & & \\ & & & & & & e^{-\lambda_2 N} & \\ & & & & & & & e^{-\lambda_1 N} \end{pmatrix}$$

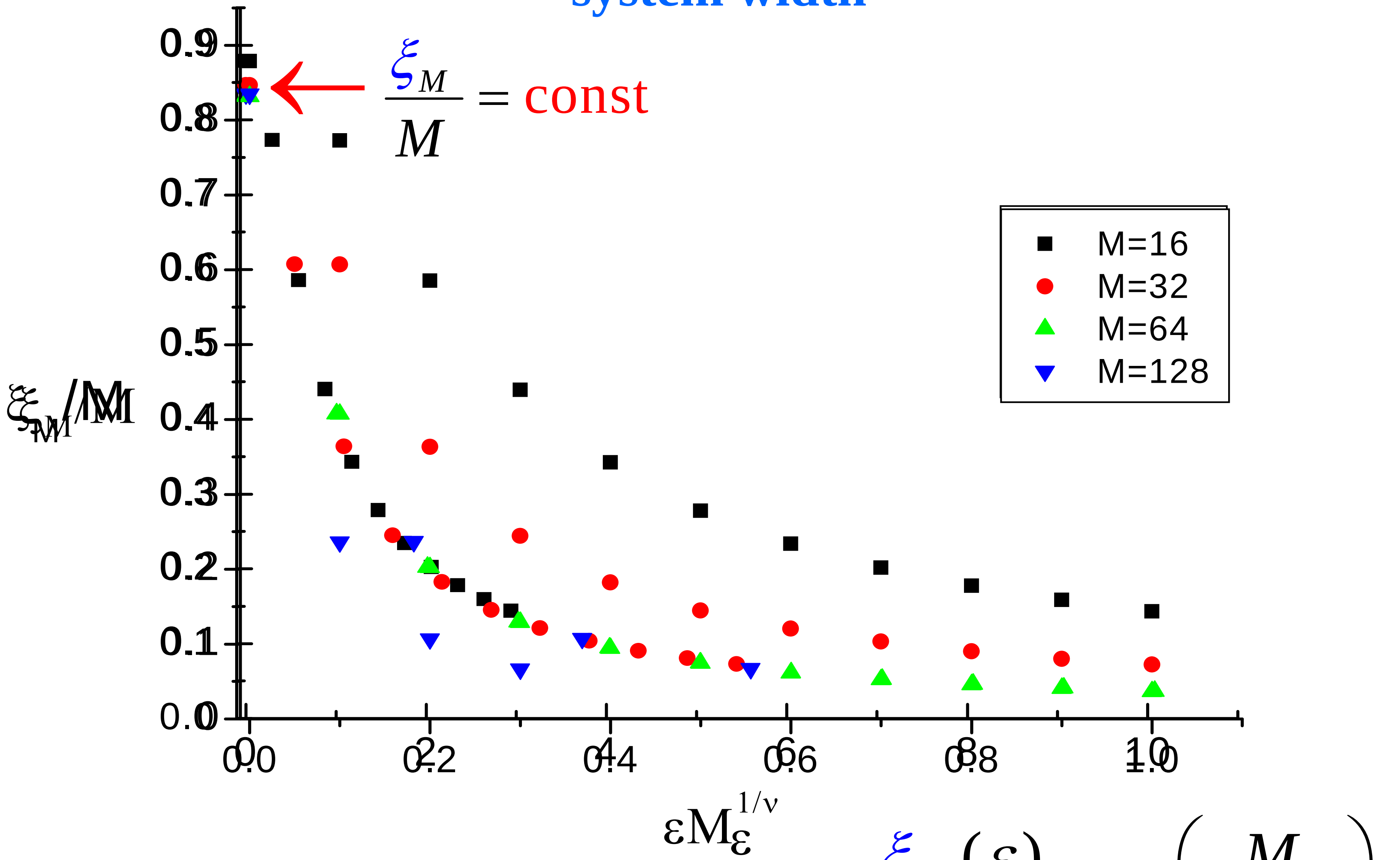
M **– system width** **Lyapunov exponents**

$$\lambda_1 > \lambda_2 > \ldots > \lambda_{M/2} > 0$$

Localization length for the system of width M ξ_M is related to the smallest positive Lyapunov exponent:

$$\xi_M \sim 1/\lambda_{M/2}$$

Renormalized localization length as function of energy and system width



$$\frac{\xi_M(\varepsilon)}{M} = f\left(\frac{M}{\xi(\varepsilon)}\right)$$

One-parameter scaling fits data for different M on one curve

The thermodynamic localization length is then defined as function of energy and diverges as energy approaches zero

$$\xi(\varepsilon) \sim |\varepsilon|^{-\nu}$$

Main result $\nu = 2.5 \pm 0.5$ **in agreement with experiment and other numerical simulations**

Is that it?

Generalization: each link carries two channels.

Mixing on the links is unitary 2x2 matrix

$$\mathrm{U} = e^{i\delta}\begin{pmatrix} e^{i\alpha}\cos\phi & e^{i\gamma}\sin\phi \\ -e^{-i\gamma}\sin\phi & e^{-i\alpha}\cos\phi \end{pmatrix}$$

Lee and Chalker, PRL 72, 1510 (1994)

Main result – two different critical energies even for the spin degenerate case

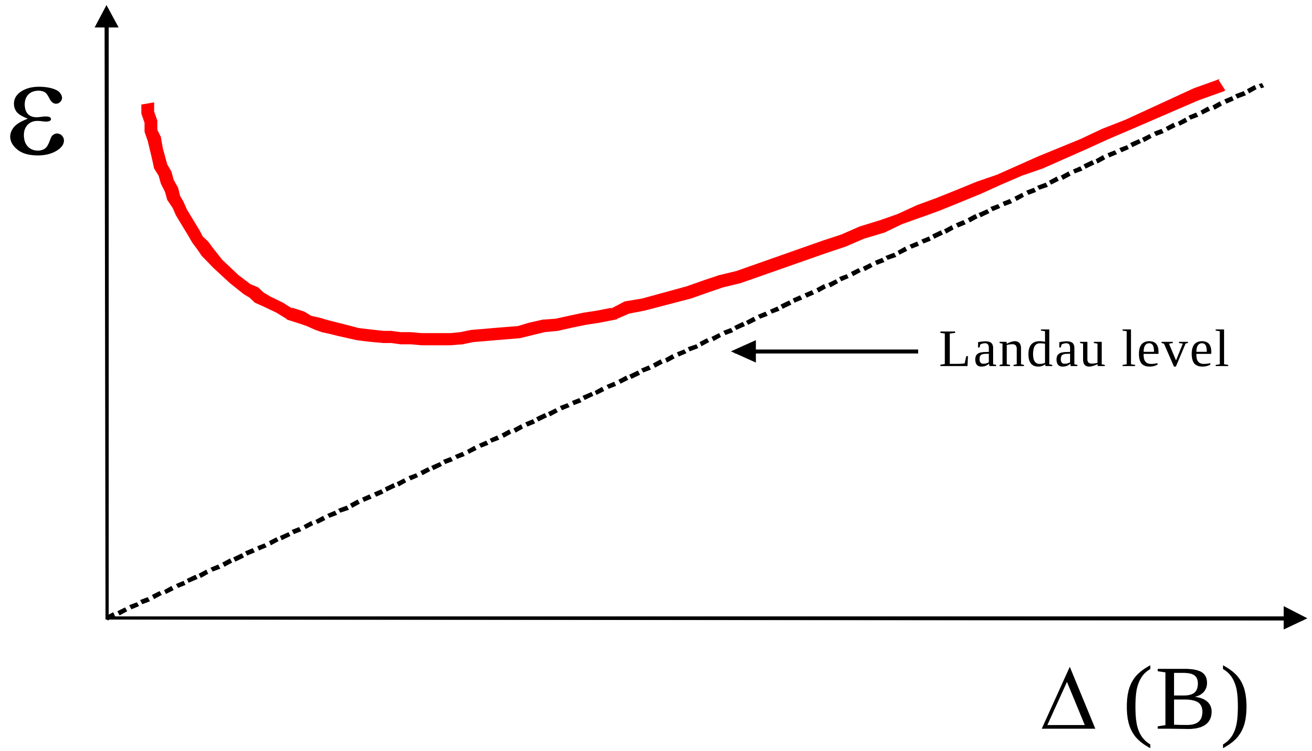
energies ε and ε-Δ, respectively

$$T = \begin{pmatrix} U_1 & \\ & U_2 \end{pmatrix} \begin{pmatrix} C & S \\ S & C \end{pmatrix} \begin{pmatrix} U_3 & \\ & U_4 \end{pmatrix}$$

$$C = \begin{pmatrix} \sqrt{1+\exp[-\pi(\varepsilon-\Delta)]} & 0 \\ 0 & \sqrt{1+\exp[-\pi\varepsilon]} \end{pmatrix} \qquad S = \begin{pmatrix} \exp[-\pi(\varepsilon-\Delta)/2] & 0 \\ 0 & \exp[-\pi\varepsilon/2] \end{pmatrix}$$

$$\Delta = E_2 / E_1 = f(B)$$

One of the results: Floating of extended states



PRB 52, R17044 (1996)

V.K., B. Horovitz and Y. Avishai

General Classification:
Altland, Zirnbauer, PRB 55 1142 (1997)

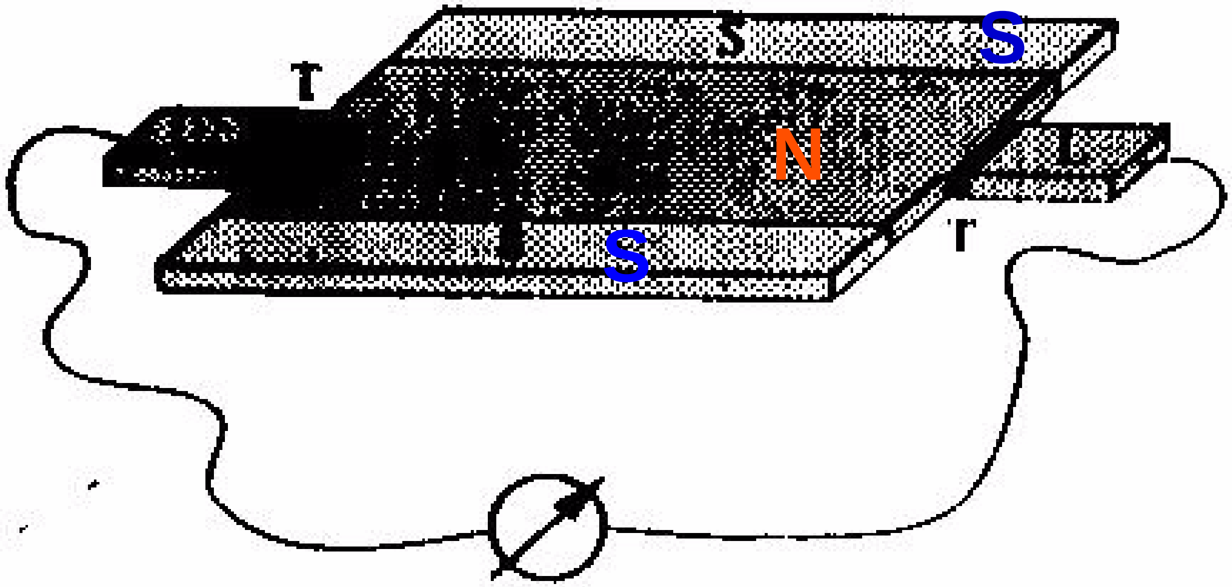


FIG. 1. Metallic quantum dot (N) in contact with two superconducting regions (S). The dot is separated from the leads (L) by a tunnel barrier (T).

Compact form of the Hamiltonian

$$\hat{H} = \frac{1}{2} \begin{pmatrix} c^\dagger & c \end{pmatrix} \begin{pmatrix} h & \Delta \\ -\Delta^* & h^T \end{pmatrix} \begin{pmatrix} c \\ c^\dagger \end{pmatrix}$$

The 4N states are arranged as (p↑,p↓,h↑,h↓)

Four additional symmetry classes: combination of time-reversal and spin-rotational symmetries

Class C – TR is broken but SROT is preserved – corresponds to SU(2) symmetry on the link in CC model (PRL 82 3516 (1999))

Renormalized localization length

$$\frac{\xi_M(\varepsilon,\Delta)}{M} = f\left(\varepsilon M^{1/\nu}, \Delta M^{1/\mu}\right)$$

with $\nu \approx 1.12, \mu \approx 1.45$

Bremen

Unidir. Motion argument

At the critical energy $$\frac{\xi_M}{M} = f\left(\varepsilon^{\nu} M, \Delta^{\mu} M\right) = \text{const}$$

and is independent of **M**, meaning the ratio between two variables is constant!

Energies of extended states $$\varepsilon_c(\Delta) = \pm c\Delta^{\mu/\nu}$$

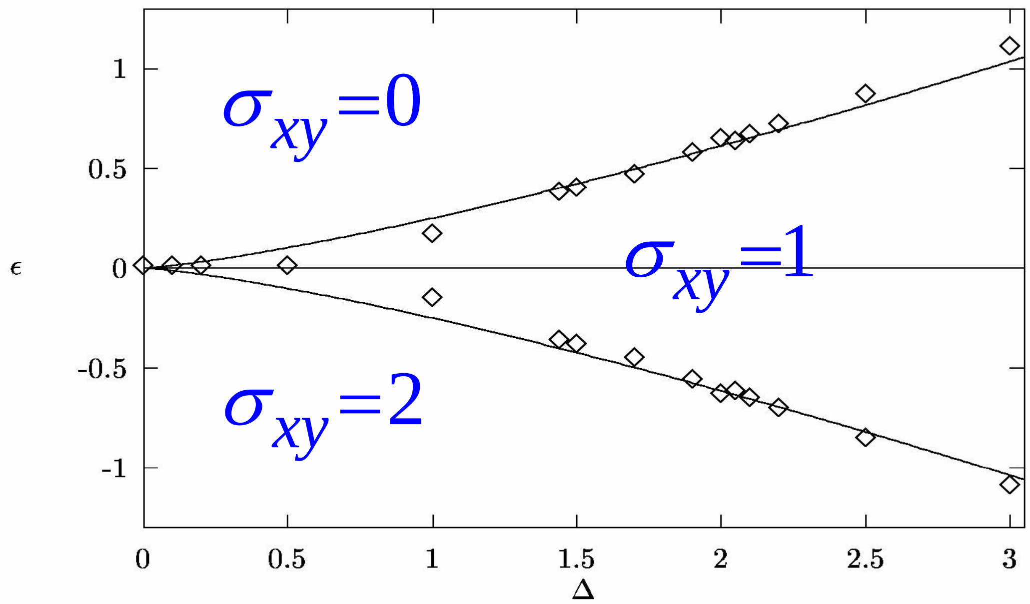


Figure 1: Phase diagram. The ◇ symbols indicate fitted positions of extended states, and the lines are $\epsilon = \pm 0.25\Delta^{1.3}$.

Spin transport

Bremen

PRL 82 3516 (1999) V.K., B. Horovitz, Y. Avishai, and J. T. Chalker

Class D – TR and SROT are broken

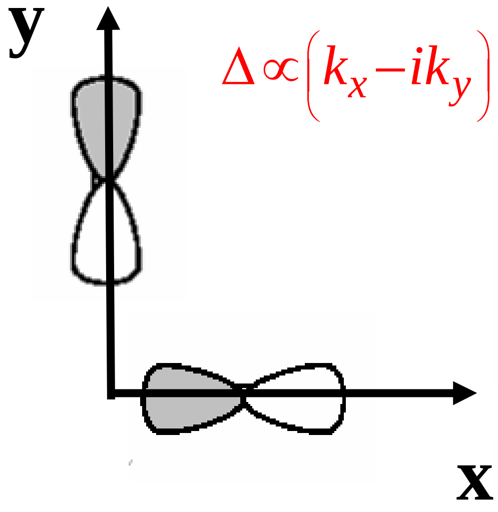
Can be realized in superconductors with a p-wave spin-triplet pairing, e.g. Sr_2RuO_4 (Strontium Ruthenate)

The A state (mixing of two different representations) – total angular momentum J_z=1 $\Rightarrow$

broken time-reversal symmetry

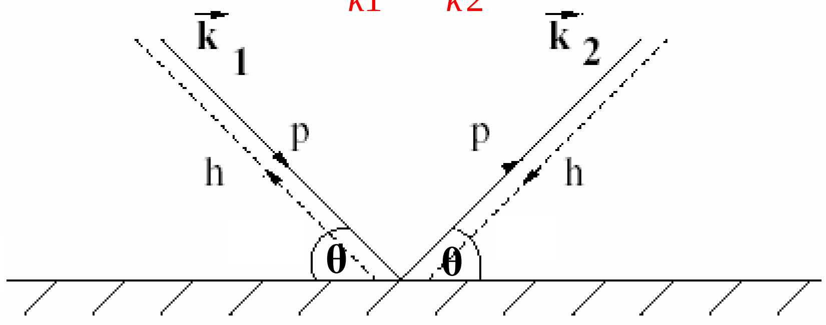
Triplet $\Rightarrow$ broken spin-rotational symmetry

$\Delta \propto (k_x - i k_y)$



p-wave

$\theta_{k1} = \theta_{k2}$



$$\Delta_{k1} = \Delta_0 \left(\cos(\theta_{k1}) - i \sin(\theta_{k1}) \right)$$

$$\Delta_{k2} = \Delta_0 \left(\cos(\theta_{k2}) + i \sin(\theta_{k2}) \right) = \Delta_0 \left(\cos(\theta_{k1}) + i \sin(\theta_{k1}) \right)$$

$\Delta_{k1} = -\Delta_{k2}$ **only for** $\theta = 90^\circ$

SNS with phase shift π $\Rightarrow$ **S | N | S**

there is a bound state

Chiral edge states imply QHE (but neither charge nor spin) – heat transport with Hall coefficient

$$K_{xy} = \frac{2\pi^2 k_B^2}{3h}$$

Bremen **Ratio** K_{xy}/T **is quantized**

Class D – TR and SROT are broken – corresponds to O(1) symmetry on the link – one-channel CC model with phases on the links (the diagonal matrix element)

$$\varphi_l = \begin{cases} 0 \text{ with probability W} \\ \pi \text{ with probability 1-W} \end{cases}$$

The result: $\lambda_M = 0$ **!!!**

M=2 exercise

$$\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^{A\theta} \begin{pmatrix} 1 \\ A \end{pmatrix}$$

After many iterations

$$\ldots = e^{[(+A+AB+ABC+\ldots)\theta]} \begin{pmatrix} 1 \\ ABC\ldots \end{pmatrix}$$

$$\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{-A\theta}\begin{pmatrix} 1 \\ -A \end{pmatrix}$$

After many iterations

$$\ldots = e^{[-(+A+AB+ABC+\ldots)\theta]}\begin{pmatrix} 1 \\ -ABC\ldots \end{pmatrix}$$

After many iterations there is a constant probability α for ABC…=+1, and correspondingly 1- α for the value -1.

Then: αW+(1- α)(1-W)= α

α=1/2 except for W=0,1

Both eigenvectors have EQUAL probability , and their contributions therefore cancel each other leading to

$$\lambda = 0$$

Change the model

$$\text{Node matrix} = \begin{pmatrix} \cosh A\theta & \sinh A\theta \\ \sinh A\theta & \cosh A\theta \end{pmatrix}$$

Cho, M. Fisher PRB 55, 1025 (1997)

Random variable A=±1 with probabilities W and 1-W respectively

$$\begin{pmatrix} \cosh A\theta & \sinh A\theta \\ \sinh A\theta & \cosh A\theta \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

Disorder in the node is equivalent to correlated disorder on the links – correlated O(1) model

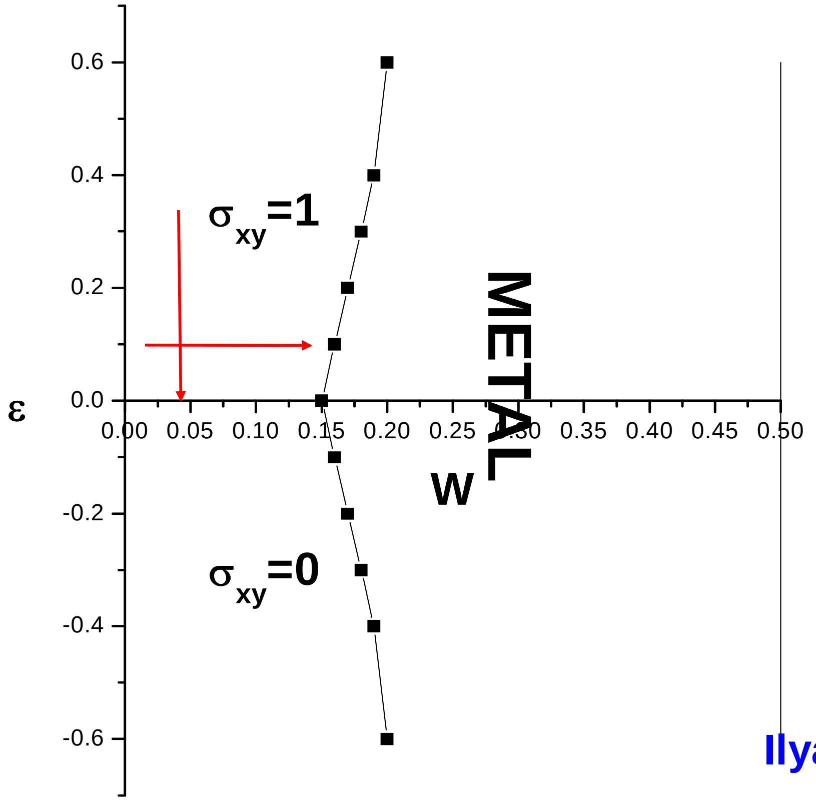
M=2 exercise

$$\begin{pmatrix} \cosh A\theta & \sinh A\theta \\ \sinh A\theta & \cosh A\theta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^{A\theta} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

λ=0 only for <A>=0, i.e. for W=1/2

Sensitivity to the disorder realization!

Heat transport



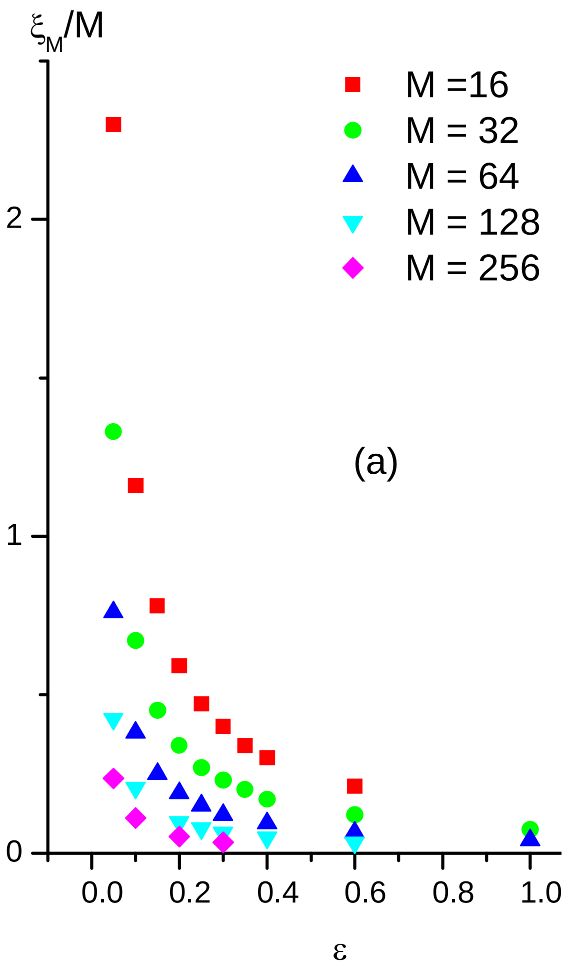
Ilya Gruzberg et. al.

PRB 65, 012506 (2001)

J. T. Chalker, N. Read, V. K., B. Horovitz, Y. Avishai, A. W. W. Ludwig:

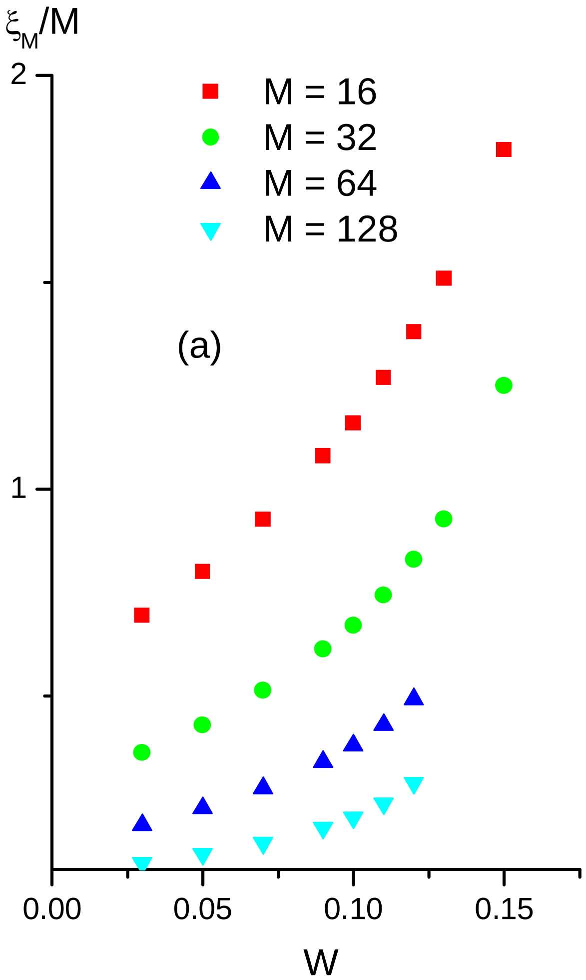
Another approach to the same problem

Another approach to the same problem



W=0.1 is fixed

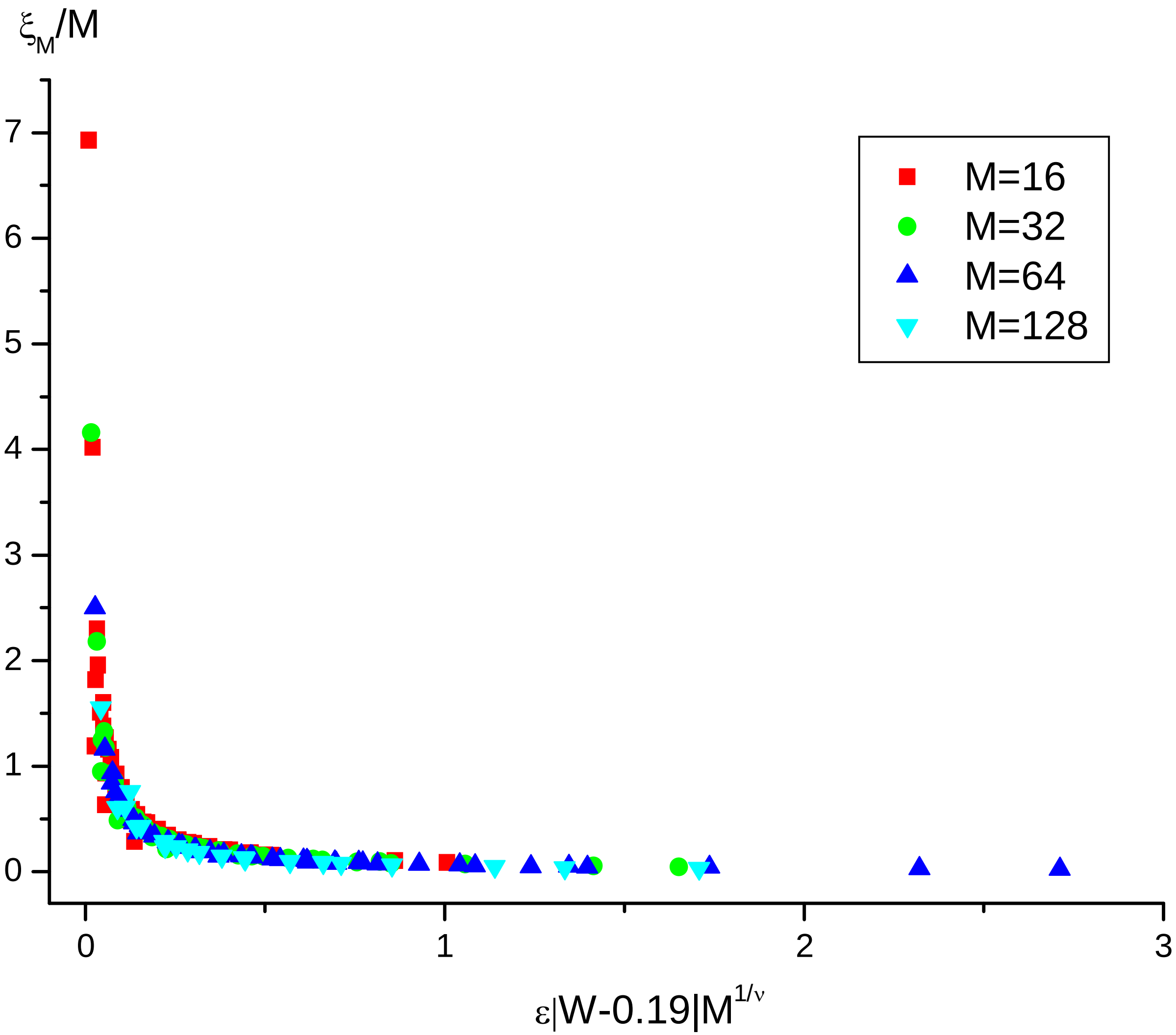
ν=1.4



ε=0.1 is fixed

ν=1.4

Bremen



ν=1.4

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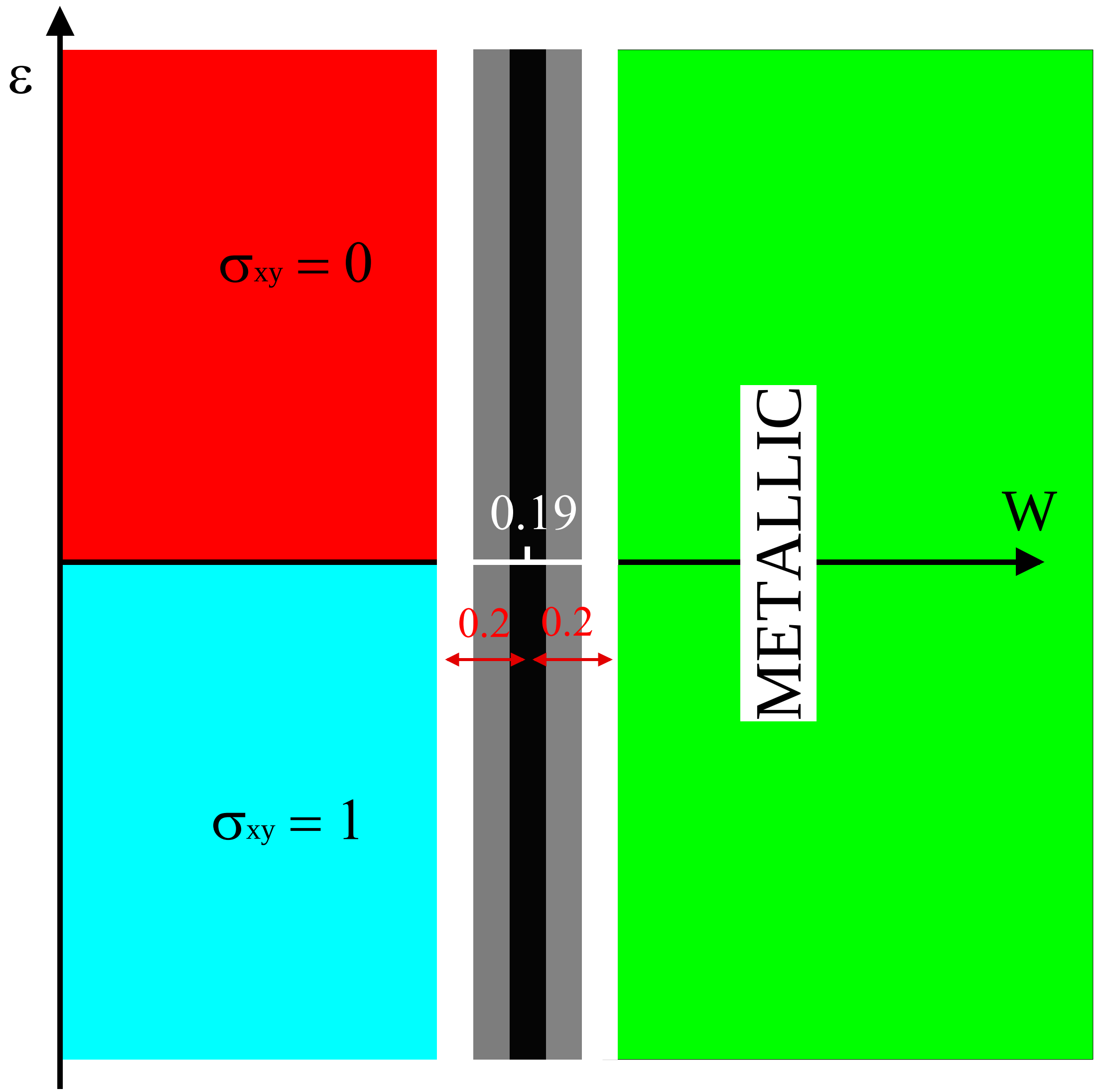


Bremen

cond-mat 0806.2744

V.K. D. Nemirovsky

Back to the original network model

Height of the barriers fluctuate - percolation

Random hyperfine fields

$$H_{\text{int}} = -\gamma_n \hbar \vec{I}_i \vec{H}_e$$

Nuclear spin

Magnetic filed produced by electrons

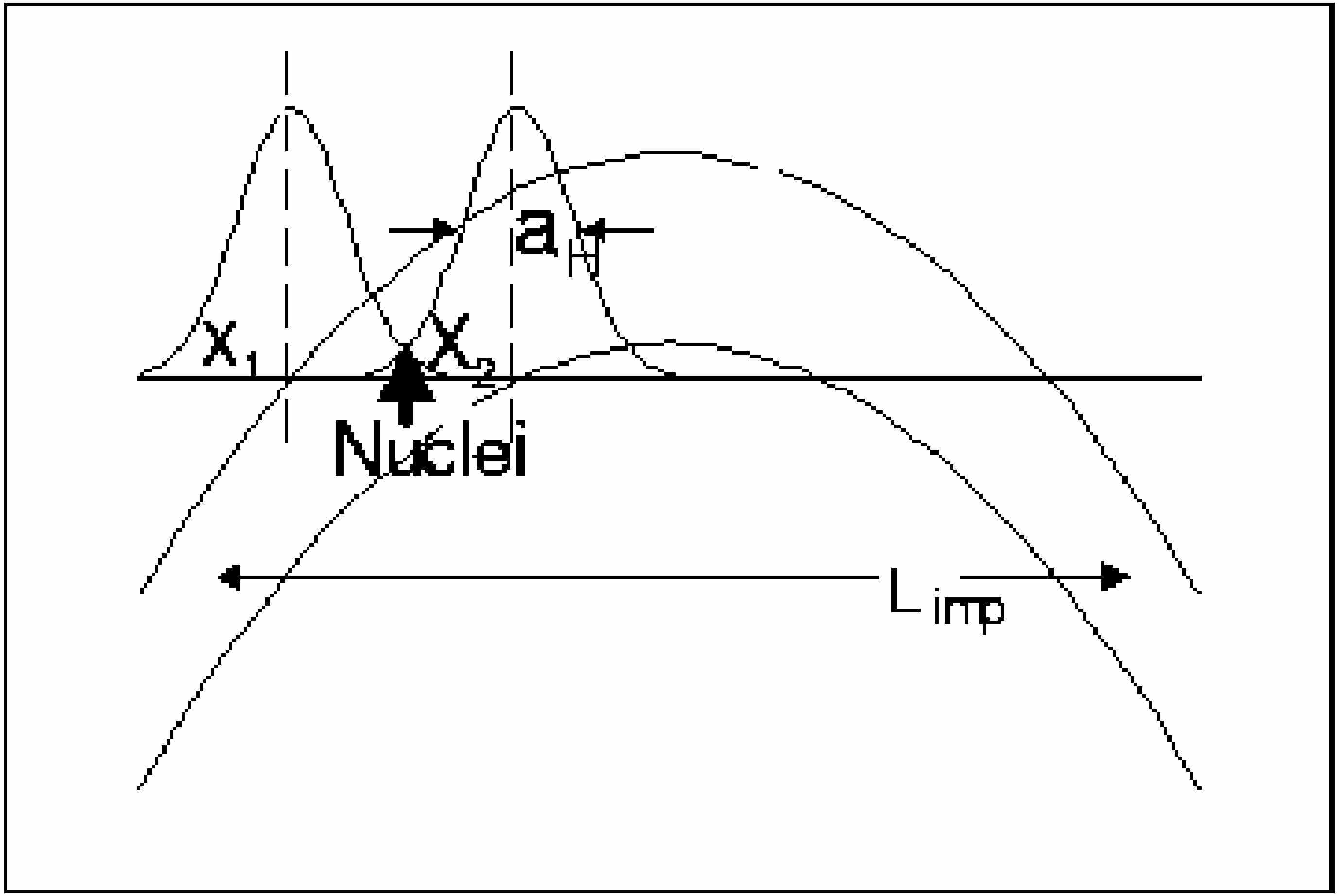
$$\vec{H}_e = -g\beta \sum_e \frac{8\pi}{3} \hat{s}_e \delta\left(\vec{r}_e - \vec{R}_i\right)$$

$$\Downarrow$$

Additional potential

$$V_{hf} = -\mu_B B_{hf}$$

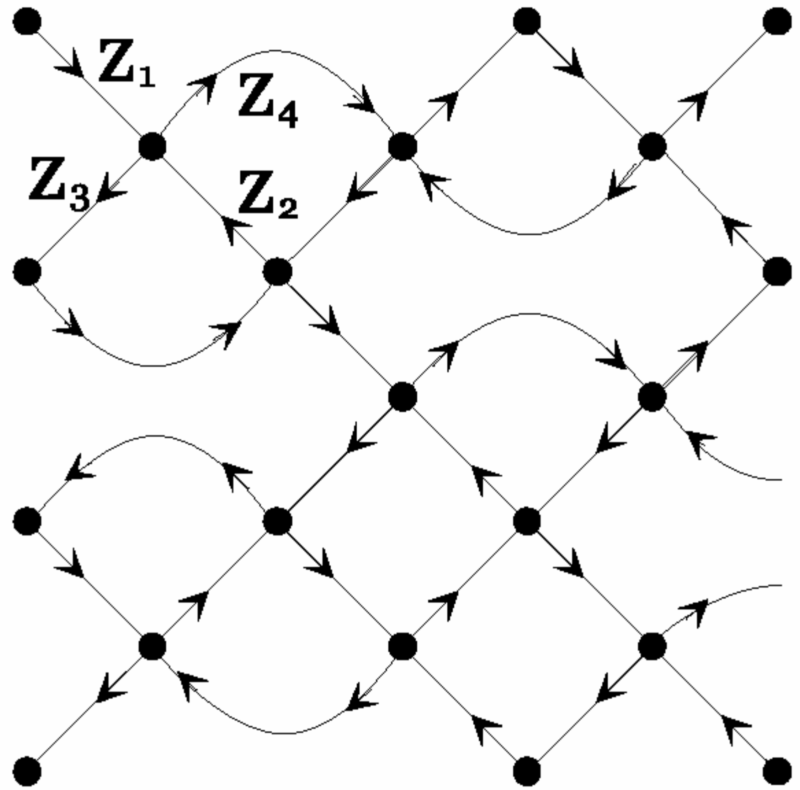
Nuclear spin relaxation



Spin-flip in the vicinity of long-range impurity

S.V. Iordanskii et. al., Phys. Rev. B **44**, 6554 (1991)
Yu.A. Bychkov et. al., Sov. Phys-JETP Lett. **33**, 143 (1981) ,

First approximation – infinite barrier with probability p

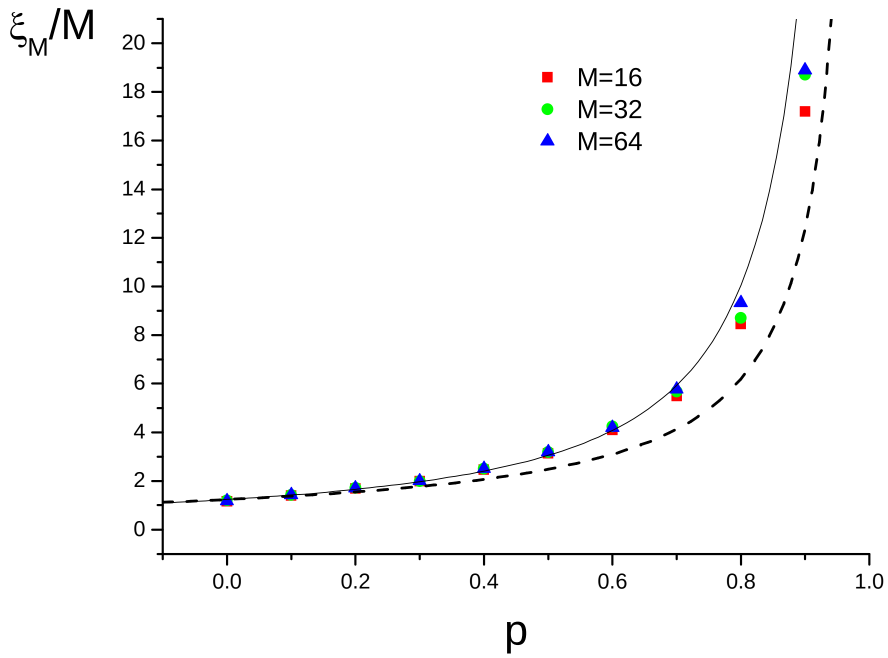


If p=1 then 2d system is broken into M 1d chains
All states are extended independent on energy
Lyapunov exponent λ=0 for any system size as in D-class superconductor

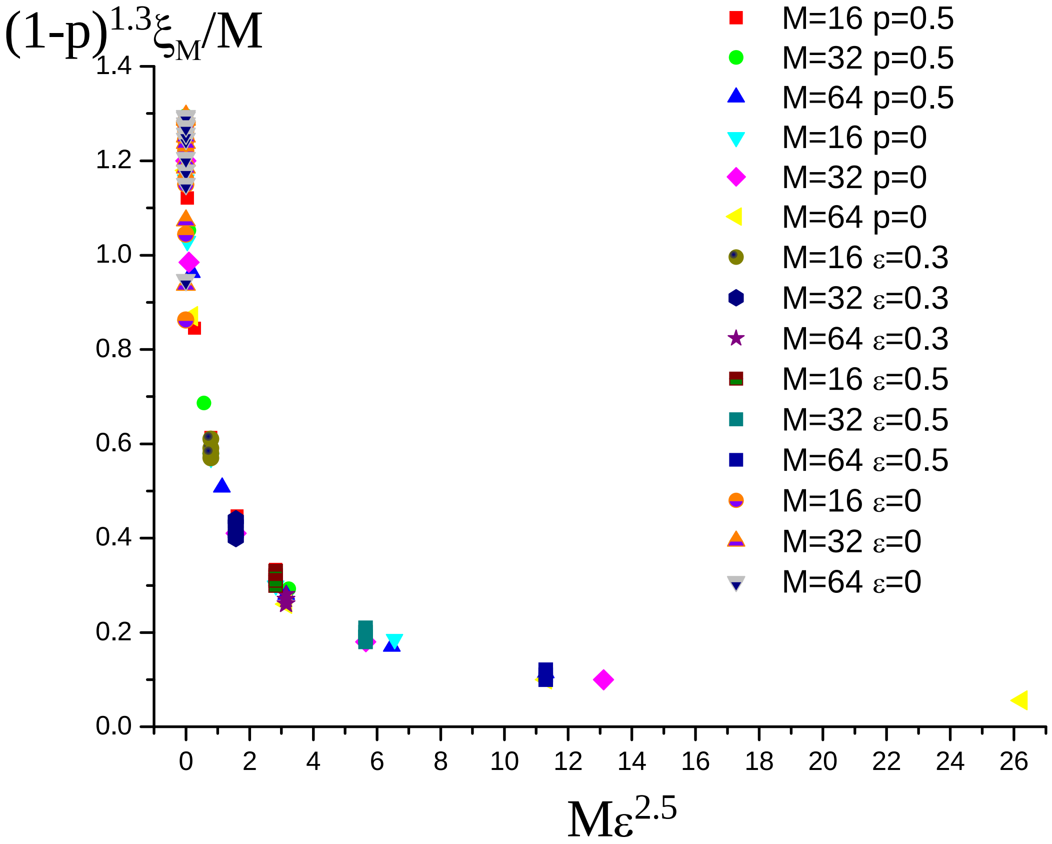
Naive argument – a fraction p of nodes is missing, therefore a particle should travel a larger distance (times 1/(1-p)) to experience the same number of scattering events, then the effective system width is $M(1-p)^{-1}$ and the scaling is

$$\frac{\xi_M}{M} = (1-p)^{-1} f\left(\frac{M}{\xi(\varepsilon)}\right)$$

But "missing" node does not allow particle to propagate in the transverse direction. Usually $\xi_M \sim M$, we, therefore, can expect power $\nu > 1$



Renormalized localization length at critical energy $\varepsilon=0$ as function of the fraction of missing nodes p for different system widths. Solid line is the best fit $1.24(1-p)^{-1.3}$. Dashed line is the fit with "naive" exponent $\nu=1$



Data collapse for all energies ε, system widths M and all fractions p≠1 of missing nodes

The effect of directed percolation can be responsible for the appearance of the value $\nu \approx 1.3$.

By making a *horizontal* direction preferential, we have introduced an anisotropy into the system.

Our result practically coincides with the value of critical exponent for the divergent temporal correlation length in 2d critical nonequilibrium systems, described by directed percolation models

H. Hinrichsen, Adv. Phys. **49**, 815 (2000)
G. Odor, Rev. Mod. Phys. **76**, 663 (2004)
S. Luebeck, Int. J. Mod. Phys. B **18**, 3977 (2004)

It probably should not come as a surprise if we recollect that each link in the network model can be associated with a unit of time

C. M. Ho and J. T. Chalker, Phys. Rev. B **54**, 8708 (1996).

Summary

Scaling

$$\frac{\xi_M}{M} = \left(1 - p\right)^{-\nu_{cl}} f\left(M\varepsilon^{\nu_q}\right)$$

$\nu_{cl} \approx 4/3$ $\nu_q \approx 2.5$

The fraction of polarized nuclei p is a relevant parameter

PRB 75, 113304 (2007) **V.K. and Israel Vagner**

Applications of Chalker-Coddington Network Model

Link Matrix	Physical System	Results	Level Statistics
U(1)	QHS: Single lowest Landau level	$\nu \approx 2.5$	GUE
U(2)	QHS: Two levels (Landau or spin) with mixing	Floating of extended states	
SU(2)	Singlet SC with SROT invariance and TR symmetry broken	Coalescence of extended states $\nu \approx 4/3$	
O(1)	Triplet SC SROT and TR symmetries broken	1D metal	GUE
Correlated O(1) - CF model		3 phase diagram: 2D metal & two insulators (σ_{xy}=0,1)	GUE for all phases?

Summary
Applications of CC network model

- QHE – one level – critical exponents
- QHE – two levels – two critical energies – floating
- QHE – current calculations
- QHE – generalization to 3d
- QHE - level statistics
- SC – spin and thermal QHE – novel symmetry classes
- SC – level statistics
- SC – 3d model for layered SC
- Chiral ensembles
- RG
- QHE and spin QHE in graphene