Finite Size Scaling

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Basis of FSS Method I

- Physical system of size $L$
- Some coupling constant $u$ (for simplicity consider only one coupling constant).
- The coupling constant is a function of some parameters $W_1, W_2$, etc. like disorder, energy etc.
- Some dimensionless physical quantity

$$Q \equiv Q(L, u(W_1, W_2, ...))$$
Renormalization Group

- Renormalization Transformation
  - Integrate out short wavelength fluctuations
  - Rescale by factor $b$ to recover same short wavelength cut-off
  - Recover original system (hopefully) but with scaled variables
- Physical quantity $Q$ now give by
  \[ Q = Q(Lb^{-1}, u') \quad u' = R_b(u) \]
- If all this makes sense then
  \[ R_{b_1b_2} = R_{b_1} R_{b_2} \]
Fixed Points

- Fixed points of RG transform correspond to phase transitions
  \[ u_c = R_b (u_c) \]

- Linearize the RG transform near the fixed point
  \[ u_c + \delta u' = u_c + \left. \frac{dR_b}{du} \right|_{u=u_c} \delta u + \cdots \]

- The derivative at the fixed point defines an exponent
  \[ b^\alpha = \left. \frac{dR_b}{du} \right|_{u=u_c} \delta u' = b^\alpha \delta u \]
Finite Size Scaling

• Use the RG transform to rescale the system to some arbitrary but fixed length \( L_0 \)

\[
Q = Q(L_0, \delta u') \quad \delta u' = b^\alpha \delta u \quad b = \frac{L}{L_0}
\]

• Since \( L_0 \) is fixed we may introduce a scaling function

\[
Q = f\left(L^\alpha \delta u\right)
\]

• This can be re-written in a more well known way

\[
Q = F_{\pm}\left(\frac{L}{\xi}\right) \quad \xi \equiv \xi(\delta u)
\]
Critical Exponent

- We usually vary one parameter at a time i.e. fix the energy and vary the disorder
- The critical point i.e. the critical value of the parameter $W$ is the solution of

$$u(W) = u_c \quad \delta u(W) = 0$$

- The correlation length diverges at the critical point

$$\xi \approx \xi_{\pm} |\delta u|^{-\nu} \quad \nu = \frac{1}{\alpha}$$
Transfer Matrices

- Consider a quasi-1D system of cross section $L \times L$ and length $L_X$.
- Transfer matrix for bar is a product of transfer matrices for each slice:
  $$M = \prod_{X=1}^{L_X} M_X$$
- The Lyapunov exponents are the limiting values of the eigenvalues of
  $$\Omega = \frac{1}{2L_X} \ln M^\dagger M \quad L_X \to \infty$$
Lyapunov Exponents

• As a consequence of current conservation the LEs occur in pairs of opposite sign.

• It is usual (but not necessary) to focus on the smallest positive exponent

\[ \gamma \equiv \gamma(L, E, W) \quad \Gamma \equiv \gamma L \]

• The output of the simulation are estimates of the LE with some precision \( \sigma \) as a function of system size \( L \), energy \( E \) and disorder \( W \)
  • Usually the LE vs \( W \) or \( E \) for a series of different \( L \)

Lyapunov Data vs System Size

- Data for Anderson model with box distributed random potential
Gamma vs System Size

- Localised Phase
  \[ \Gamma \approx \frac{L}{\xi} \quad L \to \infty \]

- Extended Phase
  \[ \Gamma \approx \frac{W^2}{L^{d-2}} \approx \frac{W^2}{L} \quad L \to \infty \]

- Scale invariance at critical point
  \[ \Gamma \approx \text{constant} \quad L \to \infty \]
Lyapunov Data vs Disorder

FSS of Lyapunovs in 3D

- We attempt to fit the data to the following form
  \[ \Gamma = f\left( L^\alpha \delta u(W) \right) \]

- Focus on the critical point: aim is to estimate the critical disorder \( W_c \) and critical exponent \( \nu \)
  \[ \delta u = w \quad w = \frac{W_c - W}{W_c} \]
  \[ f(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_n x^n \]
Least Squares Fitting

- The best fit is found by minimizing the chi-squared statistic

\[ \chi^2 = \sum_i \left( \frac{Q^{(i)} - f^{(i)}}{\sigma^{(i)}} \right)^2 \]

- Justification
  - Random normal errors in data
  - Model is non-linear in some parameters
  - Iterative solution e.g. Levenberg-Marquardt

Reliability and Precision

- It is extremely important to determine
  - the reliability of the fit
  - the precision of all estimated quantities
- Reliability of the fit
  - Goodness of fit (GOF) is the probability that we would get a worse value of chi-squared by chance
  - GOF $p > 0.1$ (or something like that)
- **Cannot proceed** unless GOF is OK
- Confidence intervals for the fitted parameters
Monte Carlo Simulation

- Generate a large series of pseudo data sets
  - Model + random normal errors
  - Fit each pseudo data set and plot histograms of chi-squared and each fitted parameter
- Goodness of fit
  - From the histogram of chi-squared
- Confidence intervals
  - Symmetric interval centred on the best fit value for the true data set and containing 95% of the data in histogram
- Alternative to MC is the Bootstrap (but no GOF)
Goodness of Fit

- Determine the goodness of fit by looking at the histogram of chi-squared

Confidence Intervals

- Determine confidence intervals from histograms of each parameter

FSS of Conductance Distribution

- Dimensionless conductance $g$ of a mesoscopic cube of side $L$
  - Sample-to-sample fluctuation of the conductance
  - FSS not applicable to $g$ directly
  - Apply to statistics such as
    \[
    \langle g \rangle \quad \exp \langle \ln g \rangle \quad \left\langle \frac{1}{g} \right\rangle
    \]
    - Or percentiles of the distribution of $g$ e.g. median of $g$

3D Anderson Model

FIG. 3. Data for the median conductance \( g = 0.5 \) percentile, together with the best fit of Eq. (8), as a function of disorder \( W \) for systems sizes in the range \( L = 6 - 18 \).
FSS of Lyapunovs in 2D

- FSS is not restricted to the critical point
- e.g. Lyapunovs in 2D

\[ \gamma L = F\left( \frac{L}{\xi(W, E)} \right) \]

- Parameterise \( F \) e.g. using splines
- Other fitting parameters are the localisation lengths for each energy and disorder

2D Anderson Model

2D Anderson Model (scaled)

Beta Function

- We can use the results of a FSS analysis to plot the beta function

\[ Q = \frac{dQ}{d \ln L} = L \frac{dQ}{dL} = \alpha f'(L^\alpha \delta u) L^\alpha \delta u \equiv \beta(Q) \]

- Parametric plot of beta function
Numerical Estimate of $\beta$ Function

- Numerical estimate of beta functions for Lyapunov exponents

Summary

- Data quality!
  - To do FSS you need high precision data for the largest systems you can manage

- Not just for critical exponents
  - Scaling functions
  - Localisation lengths
  - Beta functions
    - Use splines etc. when needed

- Precision
  - Your estimates are **meaningless** without systematic calculation of their precision

References

- **RG and FSS**

- **FSS and Anderson localisation**
  - Slevin, K. and T. Ohtsuki (1999). "Corrections to Scaling at the Anderson Transition." PRL 82 382.
  - Slevin, K., Y. Asada, et al. (2004). "Fluctuations of the Lyapunov exponent in two-dimensional disordered systems." PRB 70 054201